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EMBEDDING OF l_∞^k IN FINITE DIMENSIONAL BANACH SPACES

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ABSTRACT

Let $x_1, x_2, \cdots, x_n$ be n unit vectors in a normed space X and define $M_n = \mathrm{Ave}\{\|\sum_{i=1}^n \varepsilon_i x_i\| : \varepsilon_i = \pm 1\}$. We prove that there exists a set $A \subset \{1, \cdots, n\}$ of cardinality $|A| \geqq [\sqrt{n}/(2^7 M_n)]$ such that $\{x_i\}_{i \in A}$ is $16M_n$-isomorphic to the natural basis of l_∞^k. This result implies a significant improvement of the known results concerning embedding of l_∞^k in finite dimensional Banach spaces. We also prove that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that every normed space X_n of dimension n either contains a $(1+\varepsilon)$-isomorphic copy of l_2^m for some m satisfying $\ln \ln m \geqq \frac{1}{2} \ln \ln n$ or contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k for some k satisfying $\ln \ln k > \frac{1}{2} \ln \ln n - C(\varepsilon)$. These results follow from some combinatorial properties of vectors with ± 1 entries.

1. Introduction

Let X be a normed space and let $A = \{x_1, \cdots, x_n\}$ be a set of n unit vectors in X. Define:

$$M_n (= M_n(A)) = \mathrm{Ave}\left\{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\| : \varepsilon_i = \pm 1\right\}.$$

For positive real numbers C and ε, let $k = k(\varepsilon, C, n)$ be the maximal nonnegative integer such that if $M_n < C$ (for some set A of n unit vectors), then X contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k. It is well known (see [15] and more generally [16]) that $k(\varepsilon, C, n) \to \infty$ as $n \to \infty$ and, in fact, the same result holds even if C grows slower than any power of n. However, in modern Banach space theory we are interested in estimating the behaviour of $k(\varepsilon, C, n)$ as n grows. The original proof implies: $k(\varepsilon, C, n) \geqq C_1(\varepsilon, C) \log \log n$. In this paper we

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improve this bound significantly by proving that $k(\varepsilon, C, n) \geqq C_1 n^{C_2 \ln(1+\varepsilon)/\ln C}$, where C_1 and C_2 are absolute constants. (See Theorem 4.2.)

Another result we prove here is the following:

THEOREM 4.3. *For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for every normed space X_n of dimension n either X_n contains a $(1+\varepsilon)$-isomorphic copy of l_2^m for some m satisfying $\ln \ln m \geqq \frac{1}{2} \ln \ln n$ or X_n contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k for some k satisfying $\ln \ln k \geqq \frac{1}{2} \ln \ln n - C(\varepsilon)$.*

This result is best possible. (See the remark following Theorem 4.3.)

Theorem 4.2 deals with a special case of the following general problem: What standard subspaces (i.e. l_p^k for some $p \geqq 1$ and some integer k) or subspaces with some symmetry are embedded in span(A) provided $M_n(A)$ has a given growth?

The case $M_n \geqq Cn^{1/p}$ for some absolute constant C and $1 \leqq p < 2$ was investigated in [2], [12] and [18] and we have a lot of information about embedding of l_r^k $(1 \leqq r < 2)$ or, more generally, of subspaces with $(1+\varepsilon)$-unconditional or $(1+\varepsilon)$-symmetric basis in span$\{x_i\}_{i=1}^n$.

The situation is completely different in the case $M_n \leqq Cn^{1/q}$ for $q > 2$. Almost no quantitative information is known in this case. In this paper we deal with the extremal situation of this case arising when M_n grows slower than any power of n.

Our results follow from some combinatorial results concerning vectors with ± 1 entries. These results, which are interesting for their own sake, are described in the next section. In Section 3 we obtain some geometric applications of the combinatorial results, and in Section 4 we prove our main results.

2. The combinatorial tools

We start with some notations and definitions. Put $N = \{1, 2, \cdots, n\}$. For a set A define $B(A) = \{f \mid f : A \to \{-1, 1\}\}$, and put $\mathcal{F} = \mathcal{F}(n) = B(N)$. In this section we study extremal properties of subsets of $\mathcal{F}$.

For $\mathcal{R} \subset \mathcal{F}$ and $I \subset N$ define

$$P^I(\mathcal{R}) = \{g \in B(I) : \exists f \in \mathcal{R} \text{ such that } g = f|_I\} \qquad \text{and}$$

$$PD^I(\mathcal{R}) = \{g \in B(I) : \exists f_1, f_2 \in \mathcal{R},\ f_1|_I = f_2|_I = g \text{ and } (f_1 + f_2)|_{N-I} \equiv 0\}.$$

$\mathcal{R}$ is I-dense if $|P^I(\mathcal{R})| = 2^{|I|}$. In other words $\mathcal{R}$ is I-dense if every function in $B(I)$ is the restriction to I of a function in $\mathcal{R}$. Similarly $\mathcal{R}$ is I-doubly-dense if $|PD^I(\mathcal{R})| = 2^{|I|}$.

For $1 \leqq m < n$, let $h(m, n)$ denote the maximal cardinality of a set $\mathcal{R} \subset \mathcal{F}$ such that for any $I \subset N$, $|I| = m$, $\mathcal{R}$ is not I-doubly-dense. Our first task in this section is to estimate $h(m,n)$ (Lemmas 2.2, 2.3(i)). In fact, the following exact value of $h(m,n)$ can be proved.

THEOREM 2.1.

$$h(m,n) = \begin{cases} \displaystyle\sum_{i=0}^{(m+n-1)/2} \binom{n}{i} & \text{if } m+n \text{ is odd,} \\ \displaystyle\binom{n-1}{(m+n)/2} + \sum_{i=0}^{(m+n-2)/2} \binom{n}{i} & \text{if } m+n \text{ is even.} \end{cases}$$

Extremal examples showing that $h(m,n)$ is at least the stated number are

$$\mathcal{R} = \{f \in \mathcal{F} : |\{i \in N : f(i) = -1\}| \leqq (m+n-1)/2\} \quad \text{if } m+n \text{ is odd}$$

and

$$\mathcal{R} = \{f \in F : |\{i \in N : f(i) = -1\}| \leqq (m+n-2)/2\}$$
$$\cup \{f \in \mathcal{F} : |\{i \in N : f(i) = -1\}| = (m+n)/2 \text{ and } f(1) = 1\} \text{ if } m+n \text{ is even.}$$

Since we need here only the asymptotic behaviour of $h(m,n)$ for large n and m and since the exact determination of $h(m,n)$ when $m+n$ is even is somewhat more complicated than the odd case, we prefer to obtain here, in part (i) of Lemma 2.3, only an upper bound to $h(m,n)$. This upper bound determines $h(m,n)$ exactly when $m+n$ is odd. The exact upper bound for the even case can be proved by combining Lemma 2.2 proven below with the theorem of Hall and König [4] and the theorem of Erdös, Ko and Rado [7].

A result similar to Theorem 2.1 determining the maximum cardinality of a set $\mathcal{R} \subset \mathcal{F}$ that is *not* I-dense for every $I \subset N$, $|I| = m$ is known (see [19], [20] and [13]), and is used in Banach space theory in [17]. It seems that the difficulty arising in the solutions of these combinatorial problems is due to the fact that these problems have many extremal sets $\mathcal{R}$ that do not share a common structure. We use here the same method used in [1] to overcome this difficulty.

DEFINITION. A set $\mathcal{R} \subset \mathcal{F}$ is called monotone if $f \in \mathcal{R}$ and $g \geqq f$ imply $g \in \mathcal{R}$.

A crucial part of the proof of these two results is the reduction of the problems to the case where $\mathcal{R}$ is monotone. Once this is done there is only one extremal set $\mathcal{R}$ (up to certain obvious isomorphisms).

For $i \in I \subset N$ and $g : I \to \{-1, 1\}$ define $\bar{T}_i(g) : I \to \{-1, 1\}$ and $\underline{T}_i(g) : I \to \{-1, 1\}$ as follows:

$$(\bar{T}_i(g))(k)=\begin{cases} g(k) & \text{if } k\neq i,\\ 1 & \text{if } k=i,\end{cases}$$

and

$$(\underline{T}_i(g))(k)=\begin{cases} g(k) & \text{if } k\neq i,\\ -1 & \text{if } k=i.\end{cases}$$

LEMMA 2.2. *For every $\mathcal{R}\subset\mathcal{F}$ there exists a monotone $\mathcal{B}\subset\mathcal{F}$ such that*

(a) $|\mathcal{B}|=|\mathcal{R}|$,

and

(b) *for all $I\subset N$ if $\mathcal{R}$ is not I-doubly-dense then $\mathcal{B}$ is not I-doubly-dense.*

PROOF. Among all sets $\mathcal{B}\subset\mathcal{F}$ that satisfy (a) and (b) let $\mathcal{B}_0$ be one for which the sum

$$M(\mathcal{B})=\sum_{f\in\mathcal{B}}\sum_{i=1}^{n}f(i) \tag{1}$$

is maximal. To complete the proof we show that $\mathcal{B}_0$ is monotone.

For $1\leqq i\leqq n$ and $f\in\mathcal{B}_0$ define

$$T_i(f)=\begin{cases}\bar{T}_i(f) & \text{if } \bar{T}_i(f)\notin\mathcal{B}_0,\\ f & \text{otherwise.}\end{cases}$$

Note that if $\mathcal{B}_0$ is not monotone then $T_i(\mathcal{B}_0)\neq\mathcal{B}_0$ for some $1\leqq i\leqq n$, and that if $f\in T_i(\mathcal{B}_0)$ and $f(i)=-1$ then both f and $\bar{T}_i(f)$ belong to $\mathcal{B}_0$.

We now show that $T_i(B_0)$ satisfies (a) and (b).

(a) It is easily checked that if $f,g\in\mathcal{B}_0$, then $f\neq g$ implies $T_i(f)\neq T_i(g)$ and thus $|T_i(\mathcal{B}_0)|=|\mathcal{B}_0|=|\mathcal{R}|$.

(b) Let $I\subset N$ and suppose that $\mathcal{R}$ is not I-doubly-dense. We must show that $T_i(\mathcal{B}_0)$ is not I-doubly-dense. $\mathcal{B}_0$ is not I-doubly-dense (since it satisfies (b)), and thus there exists a function $g:I\to\{-1,1\}$ such that $g\notin PD^I(\mathcal{B}_0)$. We consider three possible cases.

Case 1. $i\notin I$

We claim that in this case $g\notin PD^I(T_i(\mathcal{B}_0))$. Suppose this is false and $g\in PD^I(T_i(\mathcal{B}_0))$. Then there are $f_1,f_2\in T_i(\mathcal{B}_0)$ such that $f_1|_I=f_2|_I=g$ and $(f_1+f_2)|_{N-I}\equiv 0$. Since $i\notin I$ $f_1(i)\neq f_2(i)$ and we may assume that $f_1(i)=-1$ and $f_2(i)=1$. Since $f_1\in T_i(\mathcal{B}_0)$ and $f_1(i)=-1$ we conclude that both f_1 and $\bar{T}_i(f_1)$ belong to $\mathcal{B}_0$. However, $f_2\in T_i(\mathcal{B}_0)$ and thus either f_2 or $\underline{T}_i(f_2)$ belong to $\mathcal{B}_0$.

Therefore either $\{f_1, f_2\} \subset \mathcal{B}_0$ or $\{\bar{T}_i(f_1), \underline{T}_i(f_2)\} \subset \mathcal{B}_0$. In both cases $g \in PD^I(\mathcal{B}_0)$ contradicting our assumption.

Case 2. $i \in I$ and $g(i) = -1$

In this case $g \notin PD^I(T_i(\mathcal{B}_0))$ since if $f \in T_i(\mathcal{B}_0)$ and $f|_I = g$ then $f(i) = -1$ and $f \in \mathcal{B}_0$. Thus the existence of $f_1, f_2 \in T_i(\mathcal{B}_0)$ that satisfy $f_1|_I = f_2|_I = g$ and $(f_1 + f_2)|_{N-I} \equiv 0$ would imply that $f_1, f_2 \in B_0$ contradicting the assumption that $g \notin PD^I(\mathcal{B}_0)$.

Case 3. $i \in I$ and $g(i) = 1$

We claim that in this case $\underline{T}_i(g) \notin PD^I(T_i(\mathcal{B}_0))$. Suppose this is false and $\underline{T}_i(g) \in PD^I(T_i(\mathcal{B}_0))$. Then there exist $f_1, f_2 \in T_i(\mathcal{B}_0)$ that satisfy $f_1|_I = f_2|_I = \underline{T}_i(g)$ and $(f_1 + f_2)|_{N-I} \equiv 0$. Since $f_1, f_2 \in T_i(\mathcal{B}_0)$ and $f_1(i) = f_2(i) = \underline{T}_i(g)(i) = -1$ it follows that $\bar{T}_i(f_1), \bar{T}_i(f_2) \in \mathcal{B}_0$. But this implies that $g \in PD^I(\mathcal{B}_0)$, a contradiction.

This completes the proof of Case 3 and shows that $T_i(\mathcal{B}_0)$ satisfies (b).

If $T_i(\mathcal{B}_0) \neq \mathcal{B}_0$ then the sum $M(\mathcal{B}_0)$ defined in (1) is strictly smaller than $M(T_i(\mathcal{B}_0))$, contradicting the choice of $\mathcal{B}_0$. Therefore $T_i(\mathcal{B}_0) = \mathcal{B}_0$ for all $1 \leqq i \leqq n$ and thus $\mathcal{B}_0$ is monotone. This completes the proof. □

LEMMA 2.3. *Suppose $\mathcal{R} \subset \mathcal{F}(n)$. For $1 \leqq m < n$ put*

$$Y(m) = \{I \subset N;\ |I| = m \text{ and } \mathcal{R} \text{ is } I\text{-doubly-dense}\}.$$

Then:

(i) *If*

$$|\mathcal{R}| > \sum_{i=0}^{[(n+m)/2]} \binom{n}{i} \tag{2}$$

then $Y(m) \neq \emptyset$ (*i.e.* $h(m,n) \leqq \sum_{i=0}^{[(n+m)/2]} \binom{n}{i}$).

(ii) *If*

$$|\mathcal{R}| \geqq \sum_{i=0}^{[n/2+\sqrt{n}]} \binom{n}{i} \tag{3}$$

then

$$|Y([\sqrt{n}])| \geqq \frac{1}{2} \cdot \binom{n}{[\sqrt{n}]}.$$

PROOF. By Lemma 2.2 we may assume that $\mathcal{R}$ is monotone. Define

$$T = T(m) = \{(I, f) : I \subset N,\ |I| = m,\ f \in \mathcal{R} \text{ and } f|_I \equiv -1\}.$$

For $J \subset N$, $|J| = m$ let

$$T_J = \{(I, f) \in T : I = J\}.$$

For $g \in \mathcal{R}$ put

$$z(g) = |\{i \in N : g(i) = -1\}|$$

and

$$T_g = \{(I, f) \in T : f = g\}.$$

Note that $|T_g| = \binom{z(g)}{m}$. Note also that if $|T_J| > \frac{1}{2} \cdot 2^{n-m}$ for some J then the function $-1 : J \to \{-1, 1\}$ belongs to $PD^I(\mathcal{R})$. This fact and the monotonicity of $\mathcal{R}$ imply that $J \in Y$.

(i) If (2) holds, then

$$|T| = \sum \{|T_f| : f \in \mathcal{R}\} = \sum \left\{ \binom{z(f)}{m} : f \in \mathcal{R} \right\}$$

$$(4) \qquad > \sum_{i=0}^{[(n+m)/2]} \binom{n}{i} \cdot \binom{i}{m} = \binom{n}{m} \cdot \sum_{i=m}^{[(n+m)/2]} \binom{n-m}{i-m}$$

$$= \binom{n}{m} \sum_{j=0}^{[(n-m)/2]} \binom{n-m}{j} \geqq \binom{n}{m} \cdot \frac{1}{2} \cdot 2^{n-m}.$$

But $|T|$ is also the sum of the $\binom{n}{m}$ $|T_J|$'s, where $J \subset N$ and $|J| = m$. Therefore $|T_J| > \frac{1}{2} \cdot 2^{n-m}$ for at least one such J and thus $Y(m) \neq \emptyset$ as needed.

(ii) If (3) holds then the same estimate as that given in (4) implies

$$|T([\sqrt{n}])| \geqq \binom{n}{[\sqrt{n}]} \cdot \frac{3}{4} \cdot 2^{n-[\sqrt{n}]}.$$

However

$$|T| = \sum \{|T_J| : J \subset N, J = [\sqrt{n}]\},$$

and since $0 \leqq |T_J| \leqq 2^{n-[\sqrt{n}]}$ for each such J the last inequality implies that $|T_J| > \frac{1}{2} \cdot 2^{n-[\sqrt{n}]}$ for at least $\frac{1}{2}\binom{n}{[\sqrt{n}]}$ different J's. This shows that $\mathcal{R}$ is J-doubly-dense for at least $\frac{1}{2}\binom{n}{[\sqrt{n}]}$ J's and completes the proof. □

The first part of Lemma 2.3 supplies an estimate of $h(m, n)$ that is used in the next section to prove the first part of Theorem 3.1. For the proof of the second (stronger) part we need some more combinatorial results.

Before stating our next lemma we need two definitions. If $I \subset N$ and $2 \leqq i \in N$ define $\bar{S}_i(I) \subset N$ as follows:

$$\bar{S}_i(I) = \begin{cases} I - i + (i-1) & \text{if } i-1 \notin I \text{ and } i \in I, \\ I & \text{otherwise.} \end{cases}$$

A family $\mathcal{G}$ of subsets of N is called a *left ball* if $I \in \mathcal{G}$ implies $\bar{S}_i(I) \in \mathcal{G}$ for each $2 \leqq i \leqq n$. The proof of the next lemma is similar to that of Lemma 2.2.[†]

LEMMA 2.4. *Let $\mathcal{G}$ be a family of m-subsets (i.e. subsets of cardinality m) of N. Then there exists a left ball $\mathcal{A}$ of m-subsets of N such that*

(a) $|\mathcal{A}| = |\mathcal{G}|$,

and

(b) *for every $1 \leqq k \leqq |\mathcal{G}|$ and every $I_1, \cdots, I_k \in \mathcal{A}$ there exist $J_1, \cdots, J_k \in \mathcal{G}$ such that*

$$\left| \bigcup_{s=1}^{k} J_s \right| \geqq \left| \bigcup_{s=1}^{k} I_s \right| .$$

PROOF. Among all families of m-subsets $\mathcal{A}$ of N that satisfy (a) and (b) let $\mathcal{A}_0$ be one for which the sum

$$L(\mathcal{A}) = \sum_{I \in \mathcal{A}} \sum_{i \in I} i \tag{5}$$

is minimal. To complete the proof we show that $\mathcal{A}_0$ is a left ball.

For $2 \leqq i \leqq n$ and $I \in \mathcal{A}_0$ define

$$S_i(I) = \begin{cases} \bar{S}_i(I) & \text{if } \bar{S}_i(I) \notin \mathcal{A}_0, \\ I & \text{otherwise.} \end{cases}$$

Note that if $\mathcal{A}_0$ is not a left ball then $S_i(\mathcal{A}_0) \neq \mathcal{A}_0$ for some $2 \leqq i \leqq n$.

We now show that $S_i(\mathcal{A}_0)$ satisfies (a) and (b).

(a) It is easily checked that if $I, J \in \mathcal{A}$ then $I \neq J$ implies $S_i(I) \neq S_i(J)$ and thus $|S_i(\mathcal{A}_0)| = |\mathcal{A}_0| = |\mathcal{G}|$.

(b) Suppose $1 \leqq k \leqq |\mathcal{G}|$ and let $S_i(I_1), \cdots, S_i(I_k)$ be k elements of $S_i(\mathcal{A}_0)$, where $I_1, \cdots, I_k \in \mathcal{A}_0$. We shall prove that there exist $J_1, \cdots, J_k \in \mathcal{A}_0$ such that

$$\left| \bigcup_{s=1}^{k} J_s \right| \geqq \left| \bigcup_{s=1}^{k} S_i(I_s) \right| . \tag{6}$$

[†] *Note added in proof.* We have recently been informed that P. Frankl had also used this lemma in [21].

This and the fact that $\mathcal{A}_0$ satisfies (b) will clearly imply that $S_i(\mathcal{A}_0)$ satisfies (b). If $i-1 \not\in \bigcup_{s=1}^k S_i(I_s)$ or if $i \not\in \bigcup_{s=1}^k S_i(I_s)$ one can easily check that we can choose $J_1 = I_1, \cdots, J_k = I_k$ in (6). The same holds also if $i-1,\ i \in \bigcup_{s=1}^k S_i(I_s)$ and $i-1 \in \bigcup_{s=1}^k I_s$. Thus we may assume that $i-1,\ i \in \bigcup_{s=1}^k S_i(I_s)$ and $i-1 \not\in \bigcup_{s=1}^k I_s$. In this case there exists t, $1 \leqq t \leqq k$, such that $i \in I_t$, $i-1 \not\in I_t$ and $\bar{S}_i(I_t) \in \mathcal{A}_0$ and there exists $r \neq t$, $1 \leqq r \leqq k$ such that $i \in I_r$, $i-1 \not\in I_r$ and $\bar{S}_i(I_r) \not\in \mathcal{A}_0$. However, we can choose here

$$\{J_1, \cdots, J_k\} = \{I_1, \cdots, I_k\} - I_t + \bar{S}_i(I_t)$$

and this set of J's will satisfy (6). This shows that $S_i(\mathcal{A}_0)$ satisfies (b).

If $S_i(\mathcal{A}_0) \neq \mathcal{A}_0$ then the sum $L(\mathcal{A}_0)$ defined in (5) is strictly larger than $L(S_i(\mathcal{A}_0))$, contradicting the choice of $\mathcal{A}_0$. Therefore $S_i(\mathcal{A}_0) = \mathcal{A}_0$ for all $2 \leqq i \leqq n$ and thus $\mathcal{A}_0$ is a left ball. This completes the proof. □

In the proof of the next lemma we use the following inequality of Chernoff (see [6]).

Chernoff's Inequality. If $0 < p < 1$, $q = 1 - p$ and k, m are integers that satisfy $k \leqq p \cdot m$ then

$$\sum_{i=0}^{k} \binom{m}{i} p^i q^{m-i} \leqq \left(\frac{mq}{m-k}\right)^{m-k} \cdot \left(\frac{mp}{k}\right)^k.$$

Lemma 2.5. *Suppose $n \geqq 1000$ and let Y be a family of $[\sqrt{n}]$-subsets of N that satisfies*

$$|Y| \geqq \frac{1}{2}\binom{n}{[\sqrt{n}]}.$$

Then there exist $k \leqq 6\sqrt{n}+1$ elements $I_1, \cdots, I_k$ of Y that satisfy

$$\left|\bigcup_{s=1}^{k} I_s\right| \geqq n - 6\sqrt{n}. \tag{7}$$

Proof. By Lemma 2.4 we may assume that Y is a left ball. In order to complete the proof we show that there exists an $I \in Y$ such that $(n - [6r\sqrt{n}] + 1) \in I$ for all $1 \leqq r < \sqrt{n}/6$. Since Y is a left ball this fact will imply the existence of $k \leqq 6\sqrt{n}+1$ elements $I_1, \cdots, I_k$ of Y that satisfy

$$\{1, 2, \cdots, n - [6\sqrt{n}] + 1\} \subset \bigcup_{s=1}^{k} I_s$$

and thus clearly satisfy (7).

Since Y is a left ball, in order to prove the existence of the desired $I \in Y$ it is enough to show that there exists a $J \in Y$ such that

$$|J \cap [n - [6r\sqrt{n}] + 1, n]| \geqq r \tag{8}$$

for all $1 \leqq r < \sqrt{n}/6$. Assume this is false. Then, for every $J \in Y$ there is some r, $1 \leqq r < \sqrt{n}/6$ such that

$$|J \cap [n - [6r\sqrt{n}] + 1, n]| < r.$$

This clearly implies

$$|Y| \leqq \sum_{1 \leqq r < \sqrt{n}/6} \sum_{i=0}^{r} \binom{[6r\sqrt{n}]}{i} \cdot \binom{n - [6r\sqrt{n}]}{[\sqrt{n}] - i}. \tag{9}$$

Since $n \geqq 1000$ one can easily check that

$$\begin{aligned} \frac{\sum_{i=0}^{r} \binom{[6r\sqrt{n}]}{i} \binom{n - [6r\sqrt{n}]}{[\sqrt{n}] - i}}{\binom{n}{[\sqrt{n}]}} &\leqq \frac{e \cdot \sum_{i=0}^{r} \frac{1}{i!([\sqrt{n}] - i)!} (6r \cdot \sqrt{n})^i \cdot (n - 6r\sqrt{n})^{[\sqrt{n}] - i}}{n^{[\sqrt{n}]}/([\sqrt{n}])!} \\ &= e \sum_{i=0}^{r} \binom{[\sqrt{n}]}{i} \left(\frac{6r}{\sqrt{n}}\right)^i \left(1 - \frac{6r}{\sqrt{n}}\right)^{[\sqrt{n}] - i}. \end{aligned} \tag{10}$$

By Chernoff's inequality the right side of (10) is at most

$$e \cdot \left(\frac{[\sqrt{n}] - 6r + 1}{[\sqrt{n}] - r}\right)^{[\sqrt{n}] - r} \left(\frac{6r}{r}\right)^r \leqq e \cdot \frac{6^r}{e^{5r-1}} = e^2 \cdot \left(\frac{6}{e^5}\right)^r.$$

Combining (9) and (10) we obtain

$$|Y| \leqq \binom{n}{[\sqrt{n}]} \cdot e^2 \sum_{1 \leqq r < \sqrt{n}/6} \left(\frac{6}{e^5}\right)^r < \frac{1}{2} \binom{n}{[\sqrt{n}]}$$

which contradicts the theorem's hypothesis. This completes the proof. □

3. Geometric applications

In this section we prove the following theorem.

THEOREM 3.1. *Let $x_1, x_2, \cdots, x_n$ be n unit vectors in a normed space and suppose*

$$M_n = \frac{1}{2^n} \sum \left\{ \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| : \varepsilon_1, \cdots, \varepsilon_n \in \{-1, 1\} \right\}.$$

Then:

(i) *There exists an* $I \subset \{1, \cdots, n\}$, $|I| = [\sqrt{n}]$ *such that*

$$\left\| \sum_{i \in I} \varepsilon_i x_i \right\| \leqq 8 \cdot M_n \tag{11}$$

for all $\varepsilon_i \in \{-1, 1\}$ $(i \in I)$.

(ii) *There exists a decomposition of* $N = \{1, 2, \cdots, n\}$ *to* $k \leqq 7\sqrt{n}$ *pairwise disjoint subsets* $I_1, \cdots, I_k$ *such that for all* $1 \leqq j \leqq k$ *and* $\varepsilon_1, \cdots, \varepsilon_n \in \{-1, 1\}$:

$$\left\| \sum_{i \in I_j} \varepsilon_i x_i \right\| \leqq 50 \cdot M_n.$$

REMARK. Both parts of Theorem 3.1 are best possible for all $1 < M_n < \sqrt{n} \cdot \sqrt{2/\pi}$ up to the constants 8 and 50. This is shown in the following example. Suppose $\sqrt{n} \geqq C > 0$ and let X be the n-dimensional (real) linear space with the norm

$$\|(y_1, \cdots, y_n)\| = \max \left\{ C \cdot \frac{\left| \sum_{i=1}^{n} y_i \right|}{\sqrt{n}}, \max_{1 \leqq i \leqq n} |y_i| \right\}.$$

Let $\{e_i\}_{i=1}^n$ be the standard basis of X. It is easily checked that the e_i's are unit vectors in X and that

$$M_n = \frac{1}{2^n} \sum \left\{ \left\| \sum_{i=1}^{n} \varepsilon_i e_i \right\| : \varepsilon_1, \cdots, \varepsilon_n \in \{-1, 1\} \right\} \leqq \sqrt{\frac{2}{\pi}}\, C + 1 + o(1).$$

However, if $I \subset \{1, \cdots, n\}$, $|I| \geqq d \cdot \sqrt{n}$ then clearly

$$\left\| \sum_{i \in I} 1 \cdot e_i \right\| \geqq d \cdot C > \sqrt{\frac{\pi}{2}}\,(M_n - 1 - o(1)).$$

PROOF. (i) It is easily checked that $M_n \geqq 1$ and thus the assertion we have to prove is trivial for $n < 80$. Thus we may assume $n \geqq 80$. Define $\mathcal{R} \subset \mathcal{F}(n)$ as follows:

$$\mathcal{R} = \left\{ f \in \mathcal{F}(n) : \left\| \sum_{i=1}^{n} f(i) \cdot x_i \right\| \leqq 8 \cdot M_n \right\}.$$

Clearly $|\mathcal{R}| \geqq \frac{7}{8} \cdot 2^n$ and since $n \geqq 80$ this implies

$$|\mathcal{R}| \geqq \sum_{i=0}^{[(n+\sqrt{n})/2]} \binom{n}{i}.$$

Therefore, by part (i) of Lemma 2.3 there exists an $I \subset N$, $|I| = [\sqrt{n}]$ such that R is I-doubly-dense. In order to complete the proof we shall show that for every $g : I \to \{-1, 1\}$

$$\left\| \sum_{i \in I} g(i) \cdot x_i \right\| \leqq 8 \cdot M_n. \tag{12}$$

For each such g there are $f_1, f_2 \in \mathcal{R}$ that satisfy $f_1|_I = f_2|_I = g$ and $(f_1 + f_2)|_{N-I} \equiv 0$. This means that

$$\sum_{i=1}^{n} g(i) \cdot x_i = \frac{1}{2} \sum_{i=1}^{n} f_1(i) x_i + \frac{1}{2} \sum_{i=1}^{n} f_2(i) \cdot x_i.$$

This, the definition of $\mathcal{R}$ and the triangle inequality imply (12) and complete the proof of part (i).

(ii) Since $M_n \geqq 1$ every $I \subset N$, $|I| = 50$ satisfies the desired inequality

$$\left\| \sum_{i \in I} \varepsilon_i x_i \right\| \leqq 50 M_n \qquad \text{for all } \varepsilon_i \in \{-1, 1\},$$

and thus the assertion of part (ii) is trivial for $n < (350)^2$. Assuming $n \geqq (350)^2$ define

$$\mathcal{R} = \left\{ f \in \mathcal{F}(n) : \left\| \sum_{i=1}^{n} f(i) \cdot x_i \right\| \leqq 50 \cdot M_n \right\}.$$

Since $n \geqq (350)^2$

$$|\mathcal{R}| \geqq \frac{49}{50} \cdot 2^n \geqq \sum_{i=0}^{[n/2 + \sqrt{n}]} \binom{n}{i}.$$

Put

$$Y = \{ I \subset N : |I| = [\sqrt{n}] \text{ and } \mathcal{R} \text{ is } I\text{-doubly-dense} \}.$$

By part (ii) of Lemma 2.3

$$|Y| \geqq \frac{1}{2} \binom{n}{[\sqrt{n}]}.$$

By Lemma 2.5 there exist $k \leqq 6\sqrt{n} + 1$ elements $J_1, \cdots, J_k$ of Y that satisfy

$$\left| \bigcup_{s=1}^{k} J_s \right| \geqq n - 6\sqrt{n}.$$

Combining double-density with the triangle inequality just as in the proof of part (i) of the theorem we conclude that for every $1 \leqq i \leqq k$, $I \subset J_i$ and $\varepsilon_j \in \{-1, 1\}$ $(j \in I)$

$$\left\|\sum_{j\in I}\varepsilon_j x_j\right\| \leqq 50\cdot M_n.$$

Clearly we can choose k pairwise disjoint sets $I_1,\cdots,I_k$ where $I_s\subset J_s$ for $1\leqq s\leqq k$ and $\bigcup_{s=1}^k J_s=\bigcup_{s=1}^k I_s$. The remaining indices lie in $I=N-\bigcup_{s=1}^k I_s$ and their number is at most $6\sqrt{n}$. We can split I into $r\leqq\sqrt{n}-1$ pairwise disjoint subsets $I_{k+1},\cdots,I_{k+r}$, each containing less than 50 indices. $I_1,I_2,\cdots,I_{k+r}$ is the desired decomposition of N. □

4. The main results

We are now ready to prove our main results. Recall that X is a normed space, $x_1,\cdots,x_n\in X$ are unit vectors and $M_n=\mathrm{Ave}\{\|\sum_{i=1}^n\varepsilon_i x_i\|:\varepsilon_i=\pm1\}$.

THEOREM 4.1. (i) *There exists a set* $A\subset\{1,2,\cdots,n\}$ *of cardinality* $k=|A|\geqq[\sqrt{n}/(2^7M_n)]$ *such that* $\{x_i\}_{i\in A}$ *is* $16M_n$*-isomorphic to the natural basis of* l_∞^k.

(ii) *There exists a decomposition of* $\{1,2,\cdots,n\}$ *to* $k\leqq 63\cdot(100M_n)^2\sqrt{n}$ *pairwise disjoint subsets* $A_1,A_2,\cdots,A_k$ *such that* $\{x_j\}_{j\in A_i}$ *is* $100M_n$*-isomorphic to* $l_\infty^{|A_i|}$ *for all* $i=1,2,\cdots,k$.

(iii) *There exists a set* $A\subset\{1,2,\cdots,n\}$ *of cardinality* $|A|\geqq[\sqrt{n}/(2^7M_n)]$ *such that* $\|\sum_{i\in A}\alpha_i x_i\|\geqq\frac{1}{2}\max_{i\in A}|\alpha_i|$ *for all* $\alpha_i\in R$. *(This shows that* $\{x_i\}_{i\in A}$ *has a biorthogonal system of functionals* $\{x_i^*\}_{i\in A}\subset X^*$ *such that* $\|x_i^*\|\leqq 2$.*) In particular, if* $M_n\leqq C\cdot n^{1/q}$ *for some* $q>2$ *then* $|A|\geqq[n^{1/2-1/q}/(2^7\cdot C)]$.

PROOF. (i) By part (i) of Theorem 3.1 there exists a set $I\subset\{1,2,\cdots,n\}$, $I=[\sqrt{n}]$ that satisfies (11). For $i\in I$ let $x_i^*\in X^*$ satisfy $\|x_i^*\|=1$ and $x_i^*(x_i)=1$. Clearly (11) implies that

$$\sum_{j\in I}|x_i^*(x_j)|\leqq 8M_n \tag{13}$$

for all $i\in I$. By [11, lemma 2] if (α_{ij}) is a nonnegative m by m matrix that satisfies $\sum_{j=1}^n\alpha_{ij}\leqq M$ for all $1\leqq i\leqq m$ then for every $\varepsilon>0$ there exists a set $J\subset\{1,\cdots,m\}$, $|J|\geqq m\cdot\varepsilon/(8M)$ such that $\sum_{j\in J\setminus\{i\}}\alpha_{ij}\leqq\varepsilon$ for all $i\in J$. Combining this lemma with (13) we obtain that for every $\varepsilon>0$ there exists $A_\varepsilon\subset I$ such that $\sum_{j\in A_\varepsilon\setminus\{i\}}|x_i^*(x_j)|<\varepsilon$ for all $i\in A_\varepsilon$ and $|A_\varepsilon|\geqq[\sqrt{n}]\cdot\varepsilon/(2^6M_n)$. By (11) for every $\alpha_i\in R$ $(i\in A_\varepsilon)$

$$\left\|\sum_{i\in A_\varepsilon}\alpha_i x_i\right\|\leqq\max_{i\in A_\varepsilon}|\alpha_i|\cdot 8M_n.$$

If $|\alpha_{i_0}|=\max_{i\in A_\varepsilon}|\alpha_i|$ then

$$\left\| \sum_{i \in A_\varepsilon} \alpha_i x_i \right\| \geqq \left| \sum_{A_\varepsilon} \alpha_i x_{i_0}^*(x_i) \right| \geqq |\alpha_{i_0}| - \sum_{i \in A_\varepsilon - \{i_0\}} |x_{i_0}^*(x_i)| \, |\alpha_{i_0}| \geqq \max_{i \in A_\varepsilon} |\alpha_i| \cdot (1 - \varepsilon).$$

To complete the proof we take $\varepsilon = 1/2$ and $A = A_{1/2}$.

(ii) The proof is analogous to that of part (i): instead of part (i) of Theorem 3.1 we use part (ii) of the theorem and instead of lemma 2 in [11] we use the following lemma of Bourgain [5]: If (α_{ij}) is a nonnegative m by m matrix that satisfies $\sum_{j=1}^{n} \alpha_{ij} \leqq M$ for all $1 \leqq i \leqq m$ then for every $\varepsilon > 0$ there exists a decomposition of $\{1, 2, \cdots, m\}$ to k pairwise disjoint subsets $A_1, A_2, \cdots, A_k$, such that $k \leqq 9 \cdot (M/\varepsilon)^2$ and for every $1 \leqq l \leqq k$ and $i \in A_l$, $\sum_{j \in A_l - \{i\}} \alpha_{ij} \leqq \varepsilon$. (In his proof Bourgain establishes the existence of two decompositions of $\{1, \cdots, m\}$ to $r \leqq 3(M/\varepsilon)$ pairwise disjoint subsets $A_1, \cdots, A_r$ and $B_1, \cdots, B_r$ such that for every $1 \leqq l \leqq r$, $i \in A_l$ and $s \in B_l$, $\sum\{\alpha_{ij} : j \in A_l, j < i\} \leqq \varepsilon/2$ and $\sum\{\alpha_{sj} : j \in B_l$ and $j > s\} \leqq \varepsilon/2$. The desired decomposition is $\{A_i \cap B_j : 1 \leqq i, j \leqq r\}$.)

(iii) The proof follows immediately from the proof of (i). □

THEOREM 4.2. *There exist absolute constants C_1 and C_2 such that for every $1 > \varepsilon > 0$, X contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k for some k satisfying $k \geqq C_1 n^{C_2 \ln(1+\varepsilon)/\ln M_n}$. In particular:*

(i) *If $M_n \leqq C$ then $k \geqq C_1 n^{C_2 \ln(1+\varepsilon)/\ln C}$.*

(ii) *There exists a constant $C(\varepsilon)$ such that if $\ln \ln M_n \leqq \delta \ln \ln n$ for some $0 < \delta < 1$ then*

$$\ln \ln k \geqq (1 - \delta) \ln \ln n - C(\varepsilon).$$

PROOF. The standard argument of [10] asserts that if $\{x_i\}_{i \in I}$ are unit vectors that satisfy

$$\left\| \sum_{i \in I} \alpha_i x_i \right\| \leqq \max |\alpha_i| \cdot C$$

for all $\alpha_i \in R$ then there exist $p = [\sqrt{|I|}]$ blocks $\{y_j\}_{j=1}^{p}$ of $\{x_i\}_{i \in I}$ that satisfy $\|y_j\| = 1$ and

$$\left\| \sum_{i=1}^{p} \alpha_i y_i \right\| \leqq \max |\alpha_i| \cdot \sqrt{C}$$

for all $\alpha_i \in R$. Combining part (i) of Theorem 3.1 with repeated applications of this argument we conclude that there exist k unit vectors $y_1, \cdots, y_k \in X$, for some k satisfying $k \geqq C_1 n^{C_2 \ln(1+\varepsilon)/\ln M_n}$ that satisfy

$$\left\| \sum_{i=1}^{k} \alpha_i y_i \right\| \leqq \max |\alpha_i| \cdot (1 + \varepsilon/3)$$

for all $\alpha_i \in R$. This and the triangle inequality imply

$$\left\| \sum_{i=1}^{k} \alpha_i y_i \right\| \geqq \max |\alpha_i| \cdot (1 - \varepsilon/3).$$

(If, e.g., $\alpha_1 = \max|\alpha_i|$ then

$$2|\alpha_1| \leqq \left\| \alpha_1 x_1 + \sum_{i=2}^{k} \alpha_i x_i \right\| + \left\| \alpha_1 x_1 - \sum_{i=2}^{k} \alpha_i x_i \right\| \leqq \left\| \sum_{i=1}^{k} \alpha_i x_i \right\| + |\alpha_1| \cdot (1 + \varepsilon/3).\Big)$$

The theorem follows since $(1+\varepsilon/3)/(1-\varepsilon/3) \leqq 1+\varepsilon$. □

COROLLARY 1. *For every $\varepsilon > 0$ and $0 < \delta < 1$ there exists a constant $C = C(\varepsilon, \delta) > 0$ such that if X_n is an n-dimensional normed space that contains a 2-isomorphic copy of l_1^m where $\ln\ln m \geqq \delta \cdot \ln\ln n$ then X_n contains a $(1+\varepsilon)$-isomorphic and $(1+\varepsilon)$-complemented copy of l_1^k where $\ln\ln k \geqq c \cdot \ln\ln n$.*

(This corollary without an estimate on k is proved in [17].)

PROOF. By the lemma in [17, section 4.1] there exist absolute positive constants C_1 and A and p functionals $\{f_i\}_{i=1}^p \subset X_n^*$ such that $\|f_i\| = 1$, $\ln\ln p \geqq C_1 \ln\ln n$ and

$$\text{Ave}\left\{ \left\| \sum_{i=1}^{p} \varepsilon_i f_i \right\|^* : \varepsilon_i = \pm 1 \right\} \leqq A(\ln n)^2.$$

This and Theorem 4.2 imply the existence of a k-dimensional space $E \subset \mathrm{sp}\{f_i\}_{i=1}^p$, where $\ln\ln k \geqq C \ln\ln n$ and E is a $(1+\varepsilon)$-isomorphic (and hence a $(1+\varepsilon)$-complemented) copy of l_∞^k. This implies the assertion of the corollary. □

THEOREM 4.3. *For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for every normed space X_n of dimension n either X_n contains a $(1+\varepsilon)$-isomorphic copy of l_2^m for some m satisfying $\ln\ln m \geqq \frac{1}{2}\ln\ln n$ or X_n contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k for some k satisfying $\ln\ln k \geqq \frac{1}{2}\ln\ln n - C(\varepsilon)$.*

PROOF. We use the results appearing in proposition 5.1 and the proof of theorem 5.2 in [9]. Let S be the ellipsoid with the maximal volume contained in the unit ball of X_n and let $|\ \ |$ be the Euclidean norm generated by this ellipsoid as the unit ball. By the Dvoretzky–Rogers Theorem there exists an orthonormal basis $\{e_i\}_{i=1}^n$ such that for $1 \leqq i \leqq k = [9n/25]$, $\|e_i\| \geqq \frac{1}{2}$. Put $M_r = (\int_S \|x\|^2 d\mu(S))^{1/2}$ where μ is the normalized Haar measure on S. It is known ([9]) that for any $\varepsilon > 0$ there exists a subspace $E \subset X_n$, such that E is $(1+\varepsilon)$-isomorphic to l_2^p and $\dim E = p \geqq B(\varepsilon) n M_r^2$ for some positive constant (depending only on ε) $B(\varepsilon)$. Define

$$\Psi(n) = B(\varepsilon) n M_r^2; \qquad M = e^{\sqrt{\ln n}}.$$

If $\Psi(n) \geqq M$ then X_n contains a $(1+\varepsilon)$-isomorphic copy of l_2^m for some $m \geqq M$ and the theorem's assertion holds. Otherwise $\Psi(n) < M$. It is well known (see, e.g., [8]) that $nM_r^2 = \int \| \sum_{i=1}^n \gamma_i e_i \|^2 d\mu(\gamma)$ where γ_i are independent normalized Gaussian variables. By [16] there exists an absolute constant C such that

$$(nM_r^2)^{1/2} = \left(\int \left\| \sum_{i=1}^n \gamma_i e_i \right\|^2 d\mu(\gamma) \right)^{1/2} \geqq \left(\int \left\| \sum_{i=1}^k \gamma_i e_i \right\|^2 d\mu(\gamma) \right)^{1/2}$$

$$\geqq \frac{1}{C \ln k} \cdot \left(\operatorname{Ave} \left\{ \left\| \sum_{i=1}^k \varepsilon_i e_i \right\|^2 : \varepsilon_i = \pm 1 \right\} \right)^{1/2} \geqq \frac{1}{C \ln k} \operatorname{Ave} \left\{ \left\| \sum_{i=1}^k \varepsilon_i e_i \right\| : \varepsilon_i = \pm 1 \right\}.$$

Therefore if $\Psi(n) \leqq M$ then

$$\operatorname{Ave} \left\{ \left\| \sum_{i=1}^k \varepsilon_i e_i \right\| : \varepsilon_i = \pm 1 \right\} \leqq C \cdot \ln k (nM_r^2)^{1/2}$$

$$= C \cdot \ln k (\Psi(n)/B(\varepsilon))^{1/2} \leqq C_1 \cdot \ln k \cdot (M/B(\varepsilon))^{1/2}.$$

For $1 \leqq i \leqq k$ put $x_i = e_i / \| e_i \|$. Since $\frac{1}{2} \leqq \| e_i \| \leqq 1$ for all $1 \leqq i \leqq k$ the last inequality clearly implies

$$\operatorname{Ave} \left\{ \left\| \sum_{i=1}^k \varepsilon_i x_i \right\| : \varepsilon_i = \pm 1 \right\} \leqq 2C_1 \cdot \ln k \cdot (M/B(\varepsilon))^{1/2}.$$

Combining this and Theorem 4.2 we conclude that X_n contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k for some k satisfying $\ln \ln k \geqq \frac{1}{2} \ln \ln n - C(\varepsilon)$ where $C(\varepsilon)$ is a constant depending only on ε. □

REMARK. It is easy to see that the estimate given in Theorem 4.3 is, in a sense, best possible. Indeed, let $X_n = l_{q_n}^n$ where $q_n = \sqrt{\ln n}$. By [3] if $E \subset X_n$ is a k-dimensional subspace which is 2-isomorphic to l_2^k then $k \leqq C n^{2/q_n}$ for some absolute constant C and thus $\ln \ln k \leqq \frac{1}{2} \ln \ln n + 1$ for sufficiently large n.

On the other hand, X_n contains no 2-isomorphic copy of l_∞^k if $\ln \ln k \geqq \frac{1}{2} \ln \ln n$. This follows from the results of [14] that states that if $E \subset l_{q_n}^2$ is a k-dimensional subspace then

$$d(E, l_2^k) \leqq k^{1/2 - 1/q_n}$$

(d denotes here the Banach–Mazur distance). This and the triangle inequality imply

$$d(E, l_\infty^k) \geqq d(l_2^k, l_\infty^k) / d(E, l_2^k) \geqq k^{1/q_n} > 2.$$

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